

# NEWTON-OKOUNKOV POLYHEDRA FOR CHARACTER VARIETIES AND CONFIGURATION SPACES

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ABSTRACT. We construct families of Newton-Okounkov bodies for the free group character varieties and configuration spaces of any connected reductive group.

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## 1. INTRODUCTION

For a commutative algebra  $A$  (for our purposes taken over  $\mathbb{C}$ ), and a valuation  $v : A \rightarrow \mathbb{Z}^M$  of rank  $M = \dim(A)$ , the image  $v(A)$  is an affine semigroup contained in a convex body  $C_v$  called the Newton-Okounkov body of  $v$ . Newton-Okounkov bodies have recently become a subject of intense study, starting with the papers of Kaveh, Khovanskii [KK] and Lazarfeld, Mustață [LM]. When  $A$  is taken to be a coordinate ring of a scheme  $X$  (e.g. projective or affine),  $C_v$  behaves like the Newton Polytope of a toric variety, providing combinatorial models from which many geometric and algebraic invariants can be computed. Newton-Okounkov bodies play directly into the geometry of  $X$  in two related ways. In the case when  $C_v$  is polyhedral, there is flat degeneration  $X \Rightarrow X_{C_v}$ , where  $X_{C_v}$  is the toric variety attached to  $C_v$ . Additionally, Harada and Kaveh have linked Newton-Okounkov bodies  $C_v$  to the study of integrable systems in  $X$ , when  $X$  satisfies some additional conditions, [HK]. This construction is also useful in combinatorics, as the set  $v(A) \subset C_v$  provides a polyhedral labelling of a basis of  $A$  which can be brought to bear when the underlying vector space of  $A$  has an enumerative meaning. With all of these applications in mind, the purpose of this paper is to construct large families of Newton-Okounkov bodies for two classes of spaces whose geometry, algebra and combinatorics are important in representation theory, the free-group character varieties and the configuration spaces for a reductive group  $G$ .

The character variety  $\mathcal{X}(\pi, G)$  associated to a finitely generated group  $\pi$  and a connected reductive group  $G$  is defined to be the moduli space of representations of  $\pi$  in  $G$ , defined as the *GIT* quotient  $\mathcal{X}(\pi, G) = \text{Hom}(\pi, G)/G$ . When  $\pi$  is the fundamental group of a smooth manifold  $M$ ,  $\mathcal{X}(\pi, G)$  is the moduli space of flat, topological principal  $G$  bundles on  $M$ . When  $M$  is taken to be a surface, the character variety  $\mathcal{X}(\pi, G)$  naturally serves as a non-commutative generalization of Teichmüller space [FG], [Go]. Stemming from these moduli interpretations, character varieties also appear as classical spaces in gauge theory, and their coordinate algebras  $\mathbb{C}[\mathcal{X}(\pi, G)]$  appear in topological quantum field theory, [Ba]. In this paper we use combinatorial elements of this field theoretic interpretation to build Newton-Okounkov bodies for  $\mathcal{X}(F_g, G)$ , where  $F_g$  is a free group.

**Theorem 1.1.** *To the following information we associate a convex polyhedral cone  $C_{\mathbf{i}}(\Gamma)$  realized as the Newton-Okounkov body of a valuation  $v_{\mathbf{i}, \Gamma}$  on  $\mathbb{C}[\mathcal{X}(F_g, G)]$ .*

- (1) *A trivalent graph  $\Gamma$  with no leaves and  $\beta_1(\Gamma) = g$ .*
- (2) *Total orderings on the non-leaf edges  $E(\Gamma)$  and non-leaf vertices  $V(\Gamma)$ .*
- (3) *A spanning tree  $\mathcal{T} \subset \Gamma$ .*
- (4) *An orientation on the edges  $\vec{e} = E(\Gamma) \setminus E(\mathcal{T})$*
- (5) *An assignment  $\mathbf{i} : V(\Gamma) \rightarrow R(w_0)$  of reduced decomposition of the longest word  $w_0$  in the Weyl group of  $G$  to each vertex  $v \in V(\Gamma)$ .*

Note that any generating set  $\{w_1, \dots, w_g\}$  of  $F_g$  defines an automorphism  $\Psi_{\vec{w}} : \mathcal{X}(F_g, G) \rightarrow \mathcal{X}(F_g, G)$ , which can then be precomposed with the maps  $\Phi_{\mathcal{T}, \vec{e}}$ . This produces a large set of filtrations on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  by pullback which carries an action by the automorphisms of  $F_g$ .

A Newton-Okounkov body  $C_v$  associated to a projective coordinate  $R_{\mathcal{L}}$  of a projective variety  $X$  with ample line bundle  $\mathcal{L}$  is naturally a cone over a compact convex body  $\bar{C}_v$ . The techniques we develop to produce the valuations  $v_{\mathbf{i}, \Gamma}$  can also be applied to construct such a compact body for another class of algebraic varieties related to the representation theory of  $G$ . Let  $\vec{\lambda} = \lambda_1, \dots, \lambda_n \in \Delta$  be dominant weights of  $G$ , and let  $P_1, \dots, P_n$  be the parabolic subgroups which respectively stabilize the highest weight vectors in the representations  $V(\lambda_1^*), \dots, V(\lambda_n^*)$ . Recall that the flag variety  $G/P_i$  has a  $G$ -linearized line bundle  $\mathcal{L}_{\lambda^*}$ , with  $H^0(G/P_i \mathcal{L}_{\lambda^*}) = V(\lambda)$  we let  $P_{\vec{\lambda}^*}(G)$  be the following diagonal *GIT* quotient.

$$(1) \quad P_{\vec{\lambda}^*}(G) = G \backslash_{\vec{\lambda}^*} \prod G/P_i$$

This is called the configuration space of  $G$ -flags associated to  $\vec{\lambda}^*$ . Our second main theorem produces a combinatorial family of polyhedral Newton-Okounkov bodies for the canonical line bundle  $\mathcal{L}_{\vec{\lambda}^*}$  on  $P_{\vec{\lambda}^*}(G)$  associated to this quotient construction.

**Theorem 1.2.** *To the following information we associate a polytope  $C_{\mathbf{i}}(\mathcal{T}, \vec{\lambda})$ , realized as the Newton-Okounkov body of a valuation  $v_{\mathbf{i}, \mathcal{T}}$  on the projective coordinate ring  $\mathbb{C}[P_{\vec{\lambda}^*}(G)] = \bigoplus_{m \geq 0} H^0(P_{\vec{\lambda}^*}(G), \mathcal{L}(m\vec{\lambda}^*))$ .*

- (1) A trivalent tree  $\mathcal{T}$  with an ordering on leaves.
- (2) A total ordering on the non-leaf edges  $E(\mathcal{T})$  and non-leaf vertices  $V(\mathcal{T})$ .
- (3) An assignment  $\mathbf{i} : V(\mathcal{T}) \rightarrow R(w_0)$  of reduced decomposition of the longest word  $w_0$  in the Weyl group of  $G$  to each vertex  $v \in V(\mathcal{T})$ .

In particular the integer points of  $C_{\mathbf{i}}(\mathcal{T}, \vec{\lambda})$  are in bijection with a basis of the invariant tensors  $(V(\lambda_1) \otimes \dots \otimes V(\lambda_n))^G = H^0(P_{\vec{\lambda}^*}(G), \mathcal{L}(\vec{\lambda}^*))$ .

The  $C_{\mathbf{i}}(\mathcal{T}, \vec{\lambda})$  are cross-sections of a cone  $C_{\mathbf{i}}(\mathcal{T})$ , which serves as a Newton-Okounkov body of an affine master configuration space  $P_n(G)$ , defined as the following affine *GIT* quotient.

$$(2) \quad P_n(G) = G \backslash (G/U)^n$$

Here  $U \subset G$  is a maximal unipotent subgroup. Any flag variety  $G/P$  with linearization  $\mathcal{L}_{\lambda^*}$  can be obtained from  $G/U$  as a right  $\lambda^*$ -linearized *GIT* quotient by a maximal torus  $T \subset G$ . Accordingly,  $P_{\vec{\lambda}^*}(G)$  is obtained from  $P_n(G)$  by a right  $T^n$  quotient. Using the same methods as in the proof of Theorem 1.1, we produce a  $T^n$  invariant valuation  $v_{\mathbf{i}, \mathcal{T}}$  on  $\mathbb{C}[P_n(G)]$  with Newton-Okounkov body  $C_{\mathbf{i}}(\mathcal{T})$ .

**1.1. Methods.** We construct the valuations  $v_{\mathbf{i}, \Gamma}$  by building filtrations on the coordinate ring  $\mathbb{C}[\mathcal{X}(F_g, G)]$  in two steps, given in Sections 2 and 3. When the associated graded algebra of a filtration is a domain, we say it is a "strong filtration". The following (almost tautological) proposition allows us to use the notions of strong filtration and valuation interchangeably.

**Proposition 1.3.** *Let  $A$  be a domain, and let  $\mathbb{Z}^M, <$  have the structure of an ordered group. The information of a strong increasing filtration  $A = \cup_{w \in \mathbb{Z}^M} F_{\leq w}$  is equivalent to a valuation  $v : A \rightarrow \mathbb{Z}^M$ .*

*Proof.* Starting with a filtration  $F$ , define  $v_F$  by  $v_F(a) = \min\{w | a \in F_{\leq w}\}$ . For a valuation  $v$  define  $F_w^v \subset A$  by  $F_w^v = \{a | v(a) \leq w\}$ . The property  $v(ab) = v(a) + v(b)$  implies that  $F_v$  is a strong filtration. Similarly,  $F$  being a strong algebra filtration implies that  $v_F(ab) = v_F(a) + v_F(b)$ . We leave it to the reader to check the rest.  $\square$

In Section 2 we build a filtration inspired from one of the applications of character varieties to gauge theory. For a maximal compact  $K \subset G$ , BF theory on an appropriately chosen triangulated manifold  $M$  is quantized by  $L^2(\mathcal{X}(F_g, K))$ , which can be identified with the coordinate ring  $\mathbb{C}[\mathcal{X}(F_g, G)]$ , see [Ba]. The states  $L^2(\mathcal{X}(F_g, K))$  are spanned by the spin diagrams of  $K$  (equivalently  $G$ ), these are defined as follows.

**Definition 1.4.** *Let  $\Gamma$  be an oriented graph, a spin diagram with topology  $\Gamma$  is the following information.*

- (1) An assignment  $\eta : E(\Gamma) \rightarrow \Delta$ , of dominant weights to the edges of  $\Gamma$ .
- (2) An assignment of  $G$ -linear maps  $\rho$  to the vertices  $v \in V(\Gamma)$  which intertwine the incoming representations  $\bigotimes_{e \rightarrow v} V(\lambda(e))$  with the outgoing representations  $\bigotimes_{v \rightarrow f} V(\lambda(f))$ .

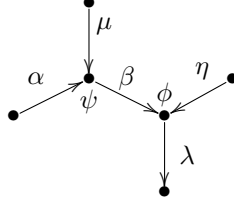


FIGURE 1. A spin diagram on a 4-tree.

The purpose of Section 2 is to show that for a fixed trivalent  $\Gamma$  with  $\beta_1(\Gamma) = g$ , the spin diagrams with topology  $\Gamma$  define a filtration of  $\mathbb{C}[\mathcal{X}(F_g, G)]$ . We identify the associated graded algebra of this filtration, and note that it is not an affine semigroup algebra unless  $G$ 's semisimple part is a product of copies of  $SL_2(\mathbb{C})$ .

In order to enhance these filtrations we must carefully choose a basis with amenable multiplication and combinatorial properties in the intertwiner spaces at each vertex  $v \in V(\Gamma)$ . This is provided by the dual canonical basis constructed by Lusztig, [Lu]. The dual canonical basis can be used to define a basis in each invariant space  $B(\mu, \lambda, \eta) \subset (V(\mu) \otimes V(\lambda) \otimes V(\eta))^G$ , which are in turn identified with intertwiner spaces, [BZ1], [BZ2]. We study filtrations built from this basis in Section 3.

For each choice  $\mathbf{i} \in R(w_0)$  of a reduced decomposition of the longest element of the Weyl group of  $G$ , there is a labelling of the dual canonical basis by tuples of non-negative integers  $b \rightarrow \vec{t} \in \mathbb{Z}^N$  called string parameters. In this way, the choice  $\mathbf{i}$  assigns the elements of  $(V(\mu) \otimes V(\lambda) \otimes V(\eta))^G$  to integer points in a convex polytope  $C_{\mathbf{i}}(\mu, \lambda, \eta)$  studied in [BZ2]. We use the inequalities of these polytopes to define the polyhedra in Theorems 1.1 and 1.2.

Our use of the dual canonical basis in this role follows previous work of Caldero [C], and Alexeev, Brion [AB], (see also Kaveh [K]), who use a filtration on the string parameters of the dual canonical basis to define full rank valuations on the coordinate rings of spherical varieties. We combine the filtrations from Section 2 with the string parameter filtrations of Section 3 with the following construction.

**Proposition 1.5.** *Let  $A$  be a domain, with  $F$  a strong filtration on  $A$  by  $\mathbb{Z}^M, <_1$ , and let  $G$  be a strong filtration on  $gr_F(A)$  by  $\mathbb{Z}^L, <_2$  which is compatible with the induced grading. There is a strong filtration  $F \circ G$  on  $A$  by  $\mathbb{Z}^{M+L}, <_1 \circ <_2$ , where  $<_1 \circ <_2$  is the composite order built lexicographically by first ordering by  $<_1$  and breaking ties with  $<_2$ . This filtration has associated graded algebra  $gr_G(gr_F(A))$ .*

*Proof.* Each space  $F_{\leq w}/F_{< w} \subset gr_F(A)$  has a filtration  $\dots \subset G_{w,u} \subset \dots$ . We pull the spaces  $G_{w,u}$  back to a filtration  $F \circ G_{w,u} \subset F_w$ . By construction each space  $F \circ G_{w,u}$  contains  $F_{< w}$ , this implies that  $F \circ G_{w',u'} \subset F \circ G_{w,u}$  if  $w' < w$ . If  $w' = w$ , then  $u' < u$  and  $F \circ G_{w',u'} \subset F \circ G_{w,u}$  by construction. It is straightforward to check the strong filtration property and the identity  $gr_{F \circ G} = gr_G(gr_F(A))$ .  $\square$

**1.2. Remarks.** Lawton [La] and Sikora [S] have given structure theorems for the coordinate rings  $\mathbb{C}[\mathcal{X}(F_g, G)]$ , and Lawton, Florentino give descriptions of the topology [FL2] and the singular locus [FL1] of  $\mathcal{X}(F_g, G)$  in certain cases. It would be interesting to relate the degenerations constructed here to a Gröbner theory of their defining equations.

Theorem 1.2 gives a construction of a basis for the tensor product invariant spaces  $(V(\lambda_1) \otimes \dots \otimes V(\lambda_n))^G$  which is labelled by the lattice points in a convex, rational polytope. Howe, Jackson, Lee, and Tan [HJLT], and Howe, Tan, Willenbring [HTW] use a SAGBI construction achieve this for triple tensor product invariant spaces in the case  $G = SL_m(\mathbb{C})$ . The cone  $\Omega_3$  resulting from their construction is a cross section of the cone of Gel'fand-Tsetlin patterns, and is linearly equivalent to  $C_1(3)$ . The algebraic structure of these cones is not very well understood outside the cases  $SL_m(\mathbb{C})$ ,  $m = 2, 3, 4$ . We also point out that the space  $P_n(SL_2(\mathbb{C}))$  is the affine cone of the Plücker embedding of the Grassmannian variety  $Gr_2(\mathbb{C}^n)$ , and that the degenerations we construct in this case coincide with those constructed by Speyer and Sturmfels in [SpSt].

Other enumeration problems in representation theory could plausibly be studied with the methods in this paper. Polyhedra which control Levi branching problems  $L \subset G$  have been defined by Berenstein and Zelevinsky in [BZ2]. These can be adapted along the lines of the program used in Sections 2, 3, and realized as Newton-Okounkov bodies.

The definition of Newton-Okounkov body we use is more general than the one in [LM] and [KK], where the valuation used to construct the Newton-Okounkov body comes from a flag of subspaces of the variety. It would be interesting to realize the tensor product polytope  $C_1(\mathcal{T}, \vec{\lambda})$  as the Newton-Okounkov body attached to a flag  $\mathcal{F}$  in a variety birational to  $P_{\vec{\lambda}}(G)$ .

The work of Harada and Kaveh [HK] suggests that each  $C_1(\mathcal{T}, \vec{\lambda})$  and  $C_1(\Gamma)$  should be the momentum image of an integrable system in  $P_{\vec{\lambda}^*}(G)$  and  $\mathcal{X}(F_g, G)$ , respectively. A construction of such an integrable system for each polyhedra would be interesting for the symplectic geometry of  $\mathcal{X}(F_g, G)$  and  $P_{\vec{\lambda}^*}(G)$ . It would also be interesting to see geometric relationships between the integrable systems associated to different valuations given by our construction. Partial results in this direction appear in [HMM] for  $G = SL_2(\mathbb{C})$ .

Finally, we remark that Theorem 1.2 essentially appears in the unpublished notes [M1], along with other remarks on the use of valuations in the study of branching problems.

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## 2. BRANCHING FILTRATIONS

In this section we make use of the ordering on dominant weights to construct filtrations of the coordinate rings of character varieties and configuration spaces. These filtrations are not fine enough to give affine semigroup associated graded algebras, however this construction reduces the problem to constructing such a filtration on  $P_3(G)$  which is  $T^3$  stable. Spin diagrams for the group  $G$  emerge from this construction as labels for the graded components of the associated graded algebras we construct. We finish the subsection with an alternative *GIT* construction of the character variety  $\mathcal{X}(F_g, G)$  which makes the connection with spin diagrams more transparent.

**2.1. Horospherical contraction and the algebra  $\mathbb{C}[G]$ .** We briefly review the theory of horospherical contraction, due to Popov [Po]. We choose a maximal torus

$T \subset G$ , with triangular decomposition  $U_-TU_+ \subset G$ , and Weyl chamber  $\Delta$ . Recall the Peter-Weyl theorem, which gives an isotypical decomposition of the vector space underlying the coordinate ring  $\mathbb{C}[G]$ .

$$(3) \quad \mathbb{C}[G] = \bigoplus_{\lambda \in \Delta} V(\lambda) \otimes V(\lambda^*)$$

We let  $b_\lambda \in V(\lambda)$  denote the highest weight vector with respect to  $U_+$ . Horospherical contraction relates  $G$  to the affine variety  $G/U$ , which has coordinate ring  $\mathbb{C}[G/U] = \bigoplus_{\lambda \in \Delta} V(\lambda) \otimes \mathbb{C}b_{\lambda^*} \subset \mathbb{C}[G]$ . Multiplication in  $\mathbb{C}[G/U]$  is computed by dualizing the map  $C : V(\lambda + \eta) \rightarrow V(\lambda) \otimes V(\eta)$  which sends  $b_{\lambda+\eta}$  to  $b_\lambda \otimes b_\eta$ .

Recall that there is a natural partial ordering on the dominant weights  $\lambda \in \Delta$ , where  $\lambda > \eta$  if  $\lambda - \eta$  can be expressed as a non-negative sum of positive roots. This ordering induces a  $G \times G$ -stable filtration on  $\mathbb{C}[G]$ .

**Proposition 2.1.** *[Horospherical contraction] The dominant weight filtration on  $\mathbb{C}[G]$  induced by  $\Delta$  has associated graded algebra isomorphic to  $\mathbb{C}[T \backslash (G/U_+ \times U_- \backslash G)]$ , where the  $T$ -action has isotypical spaces  $V(\lambda) \otimes V(\lambda^*) \subset \mathbb{C}[G/U_+ \times U_- \backslash G]$ .*

*Proof.* This follows from Chapter 3, Section 15 of [G].  $\square$

As  $\mathbb{C}[T \backslash (G/U_+ \times U_- \backslash G)]$  is a domain, any prolongation of the partial ordering on  $\Delta$  to a complete ordering which is compatible with addition of weights defines a strong filtration  $\cup_{\lambda \in \Delta} F_{\leq \lambda} = \mathbb{C}[G]$ , where  $F_{\leq \lambda} = \bigoplus_{\eta \leq \lambda} V(\eta) \otimes V(\eta^*)$ . There are many such prolongations, we define one below.

**Definition 2.2.** *Let  $G$  be a simple complex group, with Weyl chamber  $\Delta$ , and simple coweights  $H_{\alpha_1}, \dots, H_{\alpha_r}$ . The total order  $<$  is defined by lexicographically organizing the orderings defined by  $\lambda(H_{\alpha_i}) \in \mathbb{Z}$ .*

Let  $\text{Lie}(G) = \mathfrak{g} = \mathfrak{z} \oplus \bigoplus \mathfrak{g}_i$ , with  $\mathfrak{g}_i$  a simple Lie algebra, and  $\mathfrak{z}$  the Lie algebra of the center  $Z \subset G$ . The Weyl chamber  $\Delta$  has a corresponding product decomposition  $\mathfrak{z}^* \times \prod \Delta_i$ . Now we can define  $<$  on  $\Delta$  by ordering the  $\mathfrak{g}_i$  and using the induced lexicographic organization of the orderings  $<_i$ . We then break ties with any lexicographic ordering on  $\mathfrak{z}$ . The following is straightforward.

**Lemma 2.3.** *The total order  $<$  respects addition of weights, and refines the partial dominant weight ordering  $<$ .*

**2.2. Branching algebras.** We apply horospherical contraction to obtain filtrations on the coordinate rings of a class of spaces  $B(\phi)$  called branching varieties. There is one such variety for each map  $\phi : H \rightarrow G$  of complex, connected reductive groups. The space  $P_n(G)$  is recovered as the branching variety  $B(\delta_n)$ , where  $\delta_n : G \rightarrow G^{n-1}$  is the diagonal embedding. We choose maximal unipotent subgroups  $U_H \subset H$ ,  $U_G \subset G$ , and Weyl chambers  $\Delta_H, \Delta_G$ , and define  $B(\phi)$  as the following affine *GIT* quotient.

$$(4) \quad B(\phi) = H \backslash [H/U_H \times G/U_G]$$

Here the action of  $H$  is defined through  $\phi$ . The coordinate ring of  $B(\phi)$  is graded by the multiplicity spaces  $W(\mu, \lambda)$  of  $H$  irreducible representations in the irreducible representations of  $G$ , as branched over the map  $\phi$ .

$$(5) \quad V(\lambda) = \bigoplus_{\mu \in \Delta_H} W(\mu, \lambda) \otimes V(\mu)$$

$$(6) \quad \mathbb{C}[B(\phi)] = \bigoplus_{\mu, \lambda \in \Delta_H \times \Delta_G} W(\mu, \lambda)$$

The branching algebras  $\mathbb{C}[B(\phi)]$  come with special filtrations defined by diagrams in the category of reductive groups. We let  $\phi = \pi \circ \psi$  be a factorization of  $\phi$ .

$$H \xrightarrow{\psi} K \xrightarrow{\pi} G$$

The map  $\psi$  defines an action of  $H$  on  $K$ , and  $\pi$  defines an action of  $K$  on  $G$ . Using these actions, we can identify  $B(\phi)$  with the following *GIT* quotient.

$$(7) \quad B(\phi) = H \times K \backslash [H/U_H \times K \times G/U_G]$$

The space  $K \backslash K \times G/U_G$  is isomorphic to  $G/U_G$ , and likewise the resulting action of  $H$  on  $H/U_G \times G/U_G$  is induced through  $\phi = \pi \circ \psi$ . The direct sum decomposition  $\mathbb{C}[K] = \bigoplus_{\eta \in \Delta_K} V(\eta) \otimes V(\eta^*)$  induces a decomposition of  $\mathbb{C}[B(\phi)]$ .

$$(8) \quad \mathbb{C}[B(\phi)] = \bigoplus_{\lambda, \eta, \mu \in \Delta_G, \Delta_K, \Delta_H} W(\mu, \eta) \otimes W(\eta, \lambda)$$

This defines a  $T_H \times T_G$ -stable filtration  $F^{\psi, \pi}$  of  $\mathbb{C}[B(\phi)]$  by the dominant weights  $\eta \in \Delta_K$ .

$$(9) \quad F_{\leq \eta}^{\psi, \pi} = \bigoplus_{\lambda, \gamma \leq \eta, \mu} W(\lambda, \gamma) \otimes W(\gamma, \mu)$$

**Proposition 2.4.** *The associated graded algebra of  $F^{\psi, \pi}$  is the affine GIT quotient  $\mathbb{C}[T_K \backslash B(\pi) \times B(\psi)]$ , where the isotypical spaces of  $T_K$  are the  $W(\lambda, \gamma) \otimes W(\gamma, \mu)$ .*

*Proof.* The filtration  $F^{\psi, \pi}$  is induced from the horospherical filtration on  $\mathbb{C}[K]$ . The  $K \times K$  stability of horospherical contraction implies that the associated graded algebra is the domain  $\mathbb{C}[B(\pi) \times B(\psi)]^{T_K}$ . This filtration is  $T_H \times T_G$ -stable by construction.  $\square$

Notice that the associated graded algebra  $\mathbb{C}[B(\pi) \times B(\psi)]^{T_K}$  has a residual algebraic action of  $T_K$ .

We finish this subsection by applying this construction to the diagonal map  $\delta_n : G \rightarrow G^{n-1}$ . Recall that  $B(\delta_n) = P_n(G)$ . Let  $\mathcal{T}$  be a tree with two internal vertices and  $n$  leaves labelled  $0, \dots, n-1$ . Let  $k+1$  be the number of edges incident on the vertex connected to the 0 vertex, and  $m$  be the number of leaves connected to the other internal vertex. This structure defines a factorization  $\delta_n = Id^s \times \delta_m \times Id^t \circ \delta_k : G \rightarrow G^{n-1}$ , where  $s+t = k-1$ . Proposition 2.4 implies there is a filtration  $F^{\mathcal{T}}$  on  $\mathbb{C}[P_n(G)]$ , with associated graded algebra  $\mathbb{C}[P_m(G) \times P_k(G)]^{\mathcal{T}}$ .

Given a trivalent tree  $\mathcal{T}$  with  $n$  leaves, and an ordering on  $E(\mathcal{T})$ , we iterate this construction to obtain a filtration  $F^{\mathcal{T}}$  on  $\mathbb{C}[P_n(G)]$ .

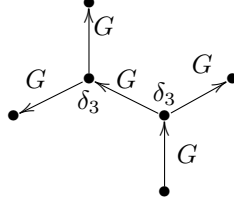


FIGURE 2. A directed tree of diagonal maps.

**Proposition 2.5.** *For a trivalent tree  $\mathcal{T}$  with  $n$  leaves, and an ordering on  $E(\mathcal{T})$  there is a  $T^n$ -invariant filtration on  $\mathbb{C}[P_n(G)]$  with associated graded algebra the coordinate ring of  $P_{\mathcal{T}}(G) = [\prod_{v \in V(\mathcal{T})} P_3(G)]/T^{E(\mathcal{T})}$ .*

*Proof.* An ordering  $E(\mathcal{T}) = \{e_1, \dots, e_{n-3}\}$  induces a length  $n-3$  chain of collapsing maps on trees,  $\pi_i : \mathcal{T}_{i-1} \rightarrow \mathcal{T}_i$  where  $\mathcal{T}_0 = \mathcal{T}$ , and  $\mathcal{T}_i$  is obtained from  $\mathcal{T}_{i-1}$  via  $\pi_i$  by collapsing the edge  $e_i$ . The tree  $\mathcal{T}_{n-3}$  has a single internal vertex, and  $\mathcal{T}_{n-4}$  has a single internal edge. By the previous construction there is a  $T^n$ -stable filtration on  $\mathbb{C}[P_n(G)]$  with associated graded algebra  $P_{\mathcal{T}_{n-4}}(G) = [P_{v(u)}(G) \times P_{v(w)}(G)]/T$ , where  $v(u)$  and  $v(w)$  are the valences of the two internal vertices  $u, w \in V(\mathcal{T}_{n-4})$ . The map  $\pi_{n-5}$  collapses the edge  $e_{n-2}$  to either  $u$  or  $w$ , yielding a corresponding filtration on  $\mathbb{C}[P_{v(u)}(G)]$  or  $\mathbb{C}[P_{v(w)}(G)]$ . This filtration is invariant with respect to  $T$  above, and so induces a filtration on  $\mathbb{C}[P_{v(u)}(G) \times P_{v(w)}(G)]/T$ . We can now apply Proposition 1.5 to obtain a filtration on  $\mathbb{C}[P_n(G)]$ . Continuing this way, we obtain the proposition.  $\square$

**2.3. Character varieties and the master configuration space.** Next we show that a similar family of filtrations can be constructed for the character variety  $\mathcal{X}(F_g, G)$ . This variety is constructed as the following *GIT* quotient.

$$(10) \quad \mathcal{X}(F_g, G) = G^g / {}_{ad}G$$

Here the *ad* subscript indicates the adjoint action of  $G$  on the product  $g \circ_{ad} (x_1, \dots, x_n) = (gx_1g^{-1}, \dots, gx_ng^{-1})$ . The coordinate ring  $\mathbb{C}[\mathcal{X}(F_g, G)]$  is therefore the algebra of adjoint  $G$  invariants in  $\mathbb{C}[G^g]$ . By Proposition 2.1, the horospherical contraction of  $\mathbb{C}[G]$  to  $\mathbb{C}[T \backslash (G/U_+ \times U_- \backslash G)]$  is  $G \times G$  invariant, therefore we may place the  $G \times G$ -stable filtration on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  to obtain the following.

**Proposition 2.6.** *There is a filtration on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  with associated graded ring equal to the coordinate ring of  $[T \backslash (G/U_+ \times U_- \backslash G)]^g / {}_{ad}G = P_{2g}(G)/T^g$ . Here the invariants of the torus  $T^g$  are the tensor products  $V(\lambda_1) \otimes V(\lambda_{2g})^G$ , where  $\lambda_{2k-1}^* = \lambda_{2k}$ .*

Now that we have connected the character variety  $\mathcal{X}(F_g, G)$  with the master configuration space  $P_{2g}(G)$ , we may use the valuations we constructed with Proposition 2.5.

**Proposition 2.7.** *For every choice of a trivalent graph  $\Gamma$ , spanning tree  $\mathcal{T} \subset \Gamma$ , an ordering on  $E(\Gamma)$  and an orientation on  $E(\Gamma) \setminus E(\mathcal{T}) = \{e_1, \dots, e_g\}$ , there is a Filtration on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  with associated graded ring the coordinate ring of  $P_{\Gamma}(G) = [\prod_{v \in V(\Gamma)} P_3(G)]/T^{E(\Gamma)}$ .*



*Proof.* We identify the ordered, oriented edges  $e_1, \dots, e_g$  with the components of  $G^g$ , where the orientation distinguishes the left and right hand sides of each component. The filtration above then yields associated graded algebra  $\mathbb{C}[P_{2g}(G)/T^g]$ . We split each edge  $e_i$  in two  $f_{2i-1}, f_{2i}$ , and build the trivalent tree  $\mathcal{T}'$  using the topology of the spanning tree  $\mathcal{T}$ . This defines a filtration on  $\mathbb{C}[P_{2g}(G)]$  with associated graded algebra  $\mathbb{C}[P_{\mathcal{T}'}(G)] = [\prod_{v \in V(\Gamma)} P_3(G)]/T^{E(\mathcal{T}')}$ . By the  $T^{2g}$  stability of the filtration and Proposition 1.5, we may now induce a filtration on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  with associated graded algebra the quotient  $P_\Gamma(G) = P_{\mathcal{T}'}(G)/T^g$ .  $\square$

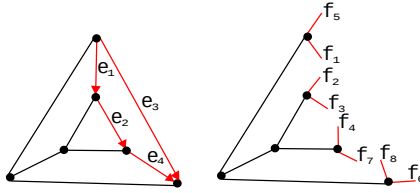


FIGURE 3. Splitting the complement of a spanning tree.

**2.4. Graph construction of character varieties.** We present an alternative construction of the variety  $\mathcal{X}(F_g, G)$ , which motivates the graph filtrations of Proposition 2.7. We fix a trivalent graph  $\Gamma$  with no leaves, and consider the forest  $\hat{\Gamma}$  obtained by splitting each edge in  $E(\Gamma)$ . This construction has also been discovered by Florentino and Lawton, [FL].

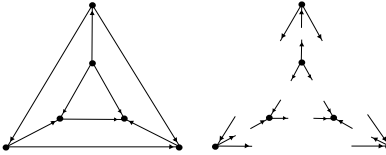


FIGURE 4. The directed forest associated to a directed graph.

We associate a copy of  $M_3(G) = G \backslash G^3$  to each connected component of  $\hat{\Gamma}$ . We then act on the product  $\prod_{v \in V(\hat{\Gamma})} M_3(G)$  with  $E(\Gamma)$  copies of  $G$ , where the component corresponding to  $e \in E(\Gamma)$  acts on the two components associated to the pair of edges in  $\hat{\Gamma}$  which split  $e$  on the right hand side. We define  $M_\Gamma(G)$  to be the *GIT* quotient by this action.

$$(11) \quad M_\Gamma(G) = [\prod_{v \in V(\Gamma)} M_3(G)]/G^{E(\Gamma)}$$

**Proposition 2.8.** *For each choice of a spanning tree  $\mathcal{T} \subset \Gamma$ , an ordering on the edges  $\vec{e} = E(\Gamma) \setminus E(\mathcal{T})$ , and an orientation of each edge in  $E(\Gamma)$ , there is an isomorphism  $\Phi_{\mathcal{T}, \vec{e}}: M_\Gamma(G) \rightarrow \mathcal{X}(F_g, G)$ .*

*Proof.* We split the edges  $e_i \in \vec{e} \subset E(\Gamma)$  into pairs  $e_{2i-1}, e_{2i}$ , ordered using the orientation on  $e_i$ , this gives a trivalent tree  $\mathcal{T}'$ . We construct  $M_{\mathcal{T}'}(G) = \prod_{v \in V(\Gamma)} M_3(G)/G^{E(\mathcal{T})}$ , and note that  $M_{\mathcal{T}'}(G)/G^{\vec{e}} = M_\Gamma(G)$ . We will prove that  $M_{\mathcal{T}'}(G) \cong G \backslash G^{2g}$  in Lemma 2.9. We use the isomorphism  $G_{2i-1} \times G_{2i}/G = G$ ,  $(g_{2i-1}, g_{2i}) \rightarrow g_{2i-1}g_{2i}^{-1}$ . This intertwines the left  $G$  action on  $G^{2g}$  with the adjoint action on  $G^g$ .  $\square$

Note that we can define  $M_\Gamma(G)$  for any graph, regardless of the valence of the vertices. In this sense we consider  $M_\mathcal{T}(G)$  for non-trivalent trees in the following lemma.

**Lemma 2.9.** *For any tree  $\mathcal{T}$  with  $n$  leaves, each orientation on the edges of  $\mathcal{T}$  gives an isomorphism to the left quotient  $M_\mathcal{T}(G) \cong G \backslash G^n$ .*

*Proof.* It suffices to treat the case where  $\mathcal{T}$  has one internal edge  $e$ , as this calculation can then be iterated to show the result by induction. Let  $\partial(e) = \{u, w\}$ , with the orientation pointing from  $u$  to  $v$ . We view  $M_\mathcal{T}(G)$  as  $G^{V(u)} \times G^{V(w)}$  with a left action of  $G \times G$  and a right action of  $G$  on two components  $G_{e,u} \subset G^{V(u)}$ ,  $G_{e,w} \subset G^{V(w)}$ . We use the map  $G^{V(u)} \times G^{V(w)} \rightarrow G^{V(u)+V(w)-2}$  given by the following.

$$(12) \quad (g_1, \dots, g_{V(u)-1}, g_{e,u}) \times (g_{e,w}, g_{V(u)+2}, \dots, g_{V(w)}) \rightarrow$$

$$(g_1, \dots, g_{V(u)-1}, g_{e,w}g_{e,u}^{-1}, g_{V(u)+2}, \dots, g_{V(w)}) \rightarrow$$

$$(g_1, \dots, g_{V(u)-1}, g_{e,u}g_{e,w}^{-1}g_{V(u)+2}, \dots, g_{e,u}g_{e,w}^{-1}g_{V(w)})$$

This is a  $G \times G \times G$ , where the first component acts diagonally on the left of  $G^{V(u)+V(w)-2}$ , and the second and third components act trivially. Quotienting everything by  $G \times G \times G$  then yields the isomorphism.  $\square$

We may also view  $M_\Gamma(G)$  as the following quotient.

$$(13) \quad \prod_{v \in V(\Gamma)} M_3(G)/G^{E(\Gamma)} = G^{V(\Gamma)} \backslash \prod_{v \in V(\Gamma)} G^3/G^{E(\Gamma)}$$

$$= G^{V(\Gamma)} \backslash \prod_{e \in E(\Gamma)} [(G \times G)/G] = G^{V(\Gamma)} \backslash G^{E(\Gamma)}.$$

We can now recover Proposition 2.7 by replacing the rightmost term with the horospherical degeneration  $G^{V(\Gamma)} \backslash \prod_{e \in E(\Gamma)} [G/U_- \times U_+ \backslash G]/T = P_\Gamma(G)$ .

### 3. VALUATIONS FROM THE DUAL CANONICAL BASIS

In the previous section we established that for each graph  $\Gamma$  (resp. tree  $\mathcal{T}$ ) with compatible information, the coordinate rings  $\mathbb{C}[\mathcal{X}(F_g, G)]$  and  $\mathbb{C}[P_n(G)]$  have the following direct sum decompositions, where  $W(\lambda, \eta, \mu)$  denotes the invariant vectors in  $V(\lambda) \otimes V(\eta) \otimes V(\mu)$ .

$$(14) \quad \mathbb{C}[\mathcal{X}(F_g, G)] = \bigoplus_{\lambda: E(\Gamma) \rightarrow \Delta} \bigotimes_{v \in V(\Gamma)} [W(\lambda(v, i), \lambda(v, j), \lambda(v, k))]$$

$$(15) \quad \mathbb{C}[P_n(G)] = \bigoplus_{\lambda: E(\mathcal{T}) \rightarrow \Delta} \bigotimes_{v \in V(\mathcal{T})} [W(\lambda(v, i), \lambda(v, j), \lambda(v, k))]$$

Here  $(v, i)$  denotes a vertex  $v$  with incident edge  $i$ . In this section we show how to obtain finer combinatorial pictures of  $\mathbb{C}[\mathcal{X}(F_g, G)]$  and  $\mathbb{C}[P_n(G)]$  by structuring the intertwiner spaces  $W(\lambda, \eta, \mu)$  using the dual canonical basis.

#### 3.1. String parameters and polytopes for tensor product multiplicities.

We recall the construction of polyhedral cones  $C_i(3)$  which control tensor product multiplicities for a reductive group  $G$ . We take  $G$  to be semisimple, but we will later remove this restriction. Lusztig [Lu] constructs a basis  $\mathbb{B}$  of the subalgebra  $\mathcal{U}_q(\mathfrak{u}_+)$  of the quantized universal enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$ . Specialization at  $q = 1$  yields the canonical basis for each irreducible representation  $V(\lambda) \subset \mathcal{U}(u_+)$ . The dual pairing between  $\mathcal{U}(u_+)$  and  $\mathbb{C}[U_+]$  induces a dual basis  $B(\lambda^*) \subset V(\lambda^*) \subset \mathbb{C}[U_+]$ . A basis  $B \subset \mathbb{C}[U_- \setminus G]$  can then be constructed by taking the union  $B = \coprod_{\lambda} B(\lambda) \times \{\lambda\}$ .

We fix a reduced decomposition  $\mathbf{i} \in R(w_0)$ . The entries of  $\mathbf{i}$  correspond to simple roots  $\alpha_{i_1}, \dots, \alpha_{i_N}$ , which in turn correspond to raising operators  $e_{i_1}, \dots, e_{i_N} \in \mathfrak{u}_+$ . These are used to define a function  $w_{\mathbf{i}} : \mathbb{C}[U_+] \rightarrow \mathbb{Z}_{\geq 0}^N$  as follows. First we compute  $t_1 = \min\{t | e_{i_1}^{t+1} \circ_{\ell} f = 0\}$ , this gives us the first  $\mathbf{i}$ -string parameter  $s_1$ , as well as a new function  $f_1 = e_{i_1}^t \circ_{\ell} f$ . We then perform the same construction with  $e_{i_2}$  and  $f_1$ , then  $e_{i_3}$  and  $f_2$ , and so on. This process produces a vector  $w_{\mathbf{i}}(f) \in \mathbb{Z}^N$ .

**Lemma 3.1.** *The function  $w_{\mathbf{i}}$  is a valuation on  $\mathbb{C}[U_+]$  when the string parameters are lex ordered first to last.*

*Proof.* By convention we set  $w_{\mathbf{i}}(0) = -\infty$ , and for any  $C \in \mathbb{C}$ ,  $w_{\mathbf{i}}(C) = 0$ . For  $f, g \in \mathbb{C}[U_+]$  let  $e_{i_k}$  be the first raising operator for which the string parameter differs, with say  $t_k(f) > t_k(g)$ . By definition,  $w_{\mathbf{i}}(f + g) = w_{\mathbf{i}}(f)$ . If no such  $k$  exists, then  $w_{\mathbf{i}}(f + g) \leq w_{\mathbf{i}}(f) = w_{\mathbf{i}}(g)$ .

Applying  $e_{i_1}^M$  to  $fg$  yields the following.

$$(16) \quad e_{i_1}^M(fg) = \sum_{p+q=M} \binom{M}{p} e_{i_1}^p(f) e_{i_1}^q(g)$$

If  $M > t_1(f) + t_1(g)$ , all terms in this sum must vanish. If  $M \leq t_1(f) + t_1(g)$ , then the fact that  $\mathbb{C}[U_+]$  is a domain implies that this sum is non-zero, as this is the case for  $M = t_1(f) + t_1(g)$ , where the sum has exactly one term, a multiple of  $e_{i_1}^{t_1(f)}(f) e_{i_1}^{t_1(g)}(g)$ . We may repeat this calculation with  $e_{i_2}$  on this term. By induction this yields  $w_{\mathbf{i}}(fg) = w_{\mathbf{i}}(f) + w_{\mathbf{i}}(g)$ .  $\square$

We obtain a valuation  $v_{\mathbf{i}}$  on the coordinate ring  $\mathbb{C}[U_- \backslash G] \subset \mathbb{C}[U_+ \times T]$ , with image in  $\mathbb{Z}^N \times \Delta$  by breaking ties in the  $<$  ordering with  $v_{\mathbf{i}}$ . Berenstein and Zelevinsky, [BZ1] and Alexeev and Brion, [AB] give inequalities for the image of this valuation using devices derived from the fundamental representations  $V(\omega_i)$  of the Langlands dual group  $\check{G}$ , called **i**-trails. An **i**-trail from a weight  $\gamma$  to a weight  $\eta$  in the weight polytope of a representation  $V$  is a sequence of weights  $(\gamma, \gamma_1, \dots, \gamma_{\ell-1}, \eta)$ , such that consecutive differences of weights are integer multiples of simple roots from  $\mathbf{i}$ ,  $\gamma_i - \gamma_{i+1} = c_k \alpha_{i_k}$ , and the application of the raising operators  $e_{i_1}^{c_1} \circ \dots \circ e_{i_{\ell}}^{c_{\ell}} : V_{\eta} \rightarrow V_{\gamma}$  is non-zero. For any **i**-trail  $\pi$ , Berenstein and Zelevinsky define  $d_k(\pi) = \frac{1}{2}(\gamma_{k-1} + \gamma_k)(H_{\alpha_{i_k}})$ . In what follows, the entries of the Cartan matrix  $A$  are denoted  $a_{ij}$ , and the element of the Weyl group  $W$  corresponding to  $\alpha_i$  is denoted  $s_i$ . For the following see [AB], [K], and [BZ1].

**Proposition 3.2.** *The image  $v_{\mathbf{i}}(B) = v_{\mathbf{i}}(\mathbb{C}[U_- \backslash G])$  is equal to the integral points in a convex polyhedral cone  $C_{\mathbf{i}} \subset \mathbb{Z}^N \times \Delta$ , defined by the following inequalities.*

- (1)  $\sum_k d_k(\pi) t_k \geq 0$  for any **i**-trail  $\omega_i \rightarrow w_0 s_i \omega_i$  in  $V(\omega_i)$ , for all fundamental weights  $\omega_j$  of the dual Langlands group.
- (2)  $t_k \leq \lambda(H_{\alpha_{i_k}}) - \sum_{\ell=k+1}^N a_{i_{\ell}, i_k} t_{\ell}$  for  $k = 1, \dots, N$ .

*Remark 3.3.* Any set  $e_1, \dots, e_m$  of  $k$ -linear nilpotent operators on a  $k$ -algebra  $A$  defines a valuation in this way. It would be very useful to know general sufficient conditions for the body  $v_{\vec{e}}(A) \subset \mathbb{R}_{\geq 0}^m$  to be polyhedral.

For  $b \in B$ , with  $v_{\mathbf{i}}(b) = (\lambda, \vec{t}) \in C_{\mathbf{i}}$  the tuple  $\vec{t} \in \mathbb{Z}^N$  is called the string parameter of  $b$  associated to  $\mathbf{i}$ . As constructed, the basis  $B$  is composed of  $T \times T$ -weight vectors, with weights  $(\sum t_i \alpha_i - \lambda, \lambda)$ . In particular,  $v_{\mathbf{i}}$  is a  $T \times T$  stable valuation. The filtration  $F^{\mathbf{i}}$  corresponding to this valuation is given by  $T \times T$ -stable subspaces  $F_{\leq (\vec{t}, \lambda)}^{\mathbf{i}} \subset \mathbb{C}[U_- \backslash G]$ , each of which has a basis of those  $b \in B$  with  $v_{\mathbf{i}}(b) \leq (\vec{t}, \lambda)$ .

Now we consider the spaces  $V_{\beta, \gamma}(\lambda) \subset V(\lambda)$ , defined as the collection of those vectors of weight  $\gamma$  which are annihilated by the raising operators  $e_i^{\beta(H_{\alpha_i})+1}$ . The basis  $B$  has the "good basis" property (see [Mat]), this implies that  $B_{\beta, \gamma}(\lambda) = B \cap V_{\beta, \gamma}(\lambda)$  is a basis for the space  $V_{\beta, \gamma}(\lambda)$ . In the case  $\beta = \eta$ , and  $\gamma = \mu^* - \eta$  this space is classically known to be isomorphic to the invariant space  $W(\mu, \lambda, \eta)$  (see [Zh], we will also provide a proof in the next subsection). Berenstein and Zelevinsky characterize the string parameters  $\vec{t}$  corresponding to the  $b \in B_{\eta, \mu^* - \eta}(\lambda)$  as follows.

**Theorem 3.4** (Berenstein, Zelevinsky, [BZ1]). *The decomposition  $\mathbf{i}$  gives a labelling of  $B_{\eta, \mu^* - \eta}(\lambda)$  by the points in  $\mathbb{Z}_{\geq 0}^N$  such that the following hold.*

- (1)  $\sum_k d_k(\pi) t_k \geq 0$  for any **i**-trail from  $\omega_j$  to  $w_0 s_j \omega_j$  in  $V(\omega_j)$ , for all fundamental weights  $\omega_j$  of the dual Langlands group.
- (2)  $-\sum_k t_k \alpha_k + \lambda + \eta = \mu^*$
- (3)  $\sum_k d_k(\pi) t_k \geq \eta(H_{\alpha_j})$  for any **i**-trail from  $s_j \omega_j$  to  $w_0 \omega_j$  in  $V(\omega_j)$ , for all fundamental weights  $\omega_j$  of the dual Langlands group.
- (4)  $t_k + \sum_{j>k} a_{i_k, i_j} t_j \geq \lambda(H_{\alpha_{i_k}})$

These are the integral points in a polytope  $C_{\mathbf{i}}(\mu, \lambda, \eta)$ .

The first and last conditions say that  $(\lambda, \vec{t})$  is a member of  $C(\mathbf{i})$  in the fiber over the weight  $\lambda$ , the second condition says that the basis members lie in the weight  $\mu^* - \eta$  subspace of  $V(\lambda)$ , and the third condition says that the appropriate raising operators  $e_i^{\eta(H_{\alpha_i})+1}$  annihilate. We realize these polytopes as slices of the following polyhedral cone.

**Definition 3.5.** For a string parameterization  $\mathbf{i}$ , the cone  $C_{\mathbf{i}}(3)$  is defined by the following inequalities on  $(\lambda, \vec{t}, \eta) \in C(\mathbf{i}) \times \Delta \subset \Delta \times \mathbb{Z}_{\geq 0}^N \times \Delta$ .

- (1)  $\sum_k d_k(\pi) t_k \geq 0$  for any  $\mathbf{i}$ -trail from  $\omega_j$  to  $w_0 s_j \omega_j$  in  $V(\omega_j)$ , for all fundamental weights  $\omega_j$  of the dual Langlands group.
- (2)  $-\sum_k t_k \alpha_k + \lambda + \eta \in \Delta$
- (3)  $\sum_k d_k(\pi) t_k \geq \eta(H_{\alpha_j})$  for any  $\mathbf{i}$ -trail from  $s_j \omega_j$  to  $w_0 \omega_j$  in  $V(\omega_j)$ , for all fundamental weights  $\omega_j$  of the dual Langlands group.
- (4)  $t_k + \sum_{j>k} a_{i_k, i_j} t_j \geq \lambda(H_{\alpha_{i_j}})$

We let  $\pi_1(\lambda, \vec{t}, \eta) = (-\sum_k t_k \alpha_k + \lambda + \eta)^*$ ,  $\pi_2(\lambda, \vec{t}, \eta) = \lambda$ , and  $\pi_3(\lambda, \vec{t}, \eta) = \eta$ . By construction  $C_{\mathbf{i}}(\mu, \lambda, \eta)$  is the fiber of these maps over  $(\mu, \lambda, \eta)$ .

**3.2. The tensor product ring map.** The  $T \times T$ -stable subspace  $\bigoplus V_{\eta, \mu^* - \eta}(\lambda) t^\eta \subset \mathbb{C}[U_- \setminus G \times T]$  inherits a basis  $B_3$  from  $B \times \Delta \subset \mathbb{C}[U_- \setminus G \times T]$ . In this subsection we show that this space is an algebra, isomorphic to  $\mathbb{C}[P_3(G)]$ . As a result we will obtain both a basis  $B_3 \subset \mathbb{C}[P_3(G)]$  and a  $T^3$ -invariant valuation  $v_{\mathbf{i}, 3} : \mathbb{C}[P_3(G)] \rightarrow C_{\mathbf{i}}(3)$  with  $v_{\mathbf{i}, 3}(B_3)$  equal to the integer points in  $C_{\mathbf{i}}(3)$ .

We use the isomorphism  $P_3(G) \cong (U_- \setminus G \times U_- \setminus G \times G/U_+)/G \cong (U_- \setminus G \times U_- \setminus G)/U_+$ . Under this map, the torus  $T^3$  which acts on  $W(\mu, \lambda, \eta) \subset \mathbb{C}[P_3(G)]$  with character  $(\mu, \lambda, \eta)$  corresponds to a torus  $T_1 \times T_2 \times T_3$  acting on  $(U_- \setminus G \times U_- \setminus G)/U_+$ . Here, the tori  $T_2$  and  $T_3$  act on the left hand sides of the components  $U_- \setminus G \times U_- \setminus G$  through the inverse, and the torus  $T_1$  acts diagonally on these components through the dual. We make use of the following commutative diagram of affine varieties.

$$\begin{array}{ccc} [T \times U_+ \times T \times U_+]/U_+ & \xleftarrow{\pi} & T \times U_+ \times T \\ \downarrow & & \downarrow \\ [U_- \setminus G \times U_- \setminus G]/U_+ & \xleftarrow{\quad} & U_- \setminus G \times T \end{array}$$

The top row is the map  $\pi : (s, u, t) \rightarrow (s, u, t, Id)$ , this is an isomorphism with inverse  $(s, x, t, y) \rightarrow (xy^{-1}, s, t)$ . Notice this that is map intertwines the left  $U_+ \times U_+$  action on  $(U_+ \times U_+)/U_+$  with the left and right actions of  $U_+$  on itself. The bottom row is defined similarly, and the vertical arrows are given by the map  $(s, u) \rightarrow su \in TU_+ \subset U_- \setminus G$ . Note that the map  $\pi$  intertwines the action of  $T_1 \times T_2 \times T_3$

We consider both the left  $\circ_\ell$  and right  $\circ_r$  actions of  $U_+$  on itself and its coordinate ring. The irreducible representation  $V(\lambda)$  has the following description as a subspace of  $\mathbb{C}[U_+]$  ([Mat], [Zh]).

$$(17) \quad V(\lambda) = \{f \in \mathbb{C}[U_+] \mid e_i^{\lambda(H_{\alpha_i})+1} \circ_\ell f = 0\}$$

Here  $1 \in \mathbb{C}[U_+]$  is identified with the highest weight vector  $v_\lambda \in V(\lambda)$ . We let  $V_\eta(\lambda) \subset V(\lambda)$  denote the space of functions  $f$  which satisfy  $e_i^{\eta(H_{\alpha_i})+1} \circ_r f = 0$ .

**Lemma 3.6.** *The following diagram commutes, and the top row is an isomorphism of vector spaces.*

$$\begin{array}{ccc} (V(\lambda) \otimes V(\eta))^{U_-} & \longrightarrow & V_\eta(\lambda) \\ \downarrow & & \downarrow \\ \mathbb{C}[(T \times U_+ \times T \times U_+)/U_+] & \xrightarrow{\pi^*} & \mathbb{C}[T \times U_+ \times T] \end{array}$$

*Proof.* We take a function  $f \in V(\lambda) \otimes V(\eta) \subset \mathbb{C}[(T \times U_+ \times T \times U_+)/U_+]$  and analyze the pullback  $\pi^*(f)$ . The function  $f$  satisfies the equations  $e_i^{\lambda(H_{\alpha_i})+1} \circ_\ell f = 0$  in the first  $U_+$  component and  $e_i^{\eta(H_{\alpha_i})+1} \circ_\ell f = 0$  in the second. By definition of  $\pi$ , these equations are satisfied if and only if  $e_i^{\lambda(H_{\alpha_i})+1} \circ_\ell \pi^*(f) = 0$ , and  $e_i^{\eta(H_{\alpha_i})+1} \circ_r \pi^*(f) = 0$ .  $\square$

Now we consider what happens when  $f \in (V(\lambda) \otimes V(\eta))^{U_+}$  has weight  $\mu^*$ , this is the case when  $f$  represents an intertwiner  $V(\mu^*) \rightarrow V(\lambda) \otimes V(\eta)$ . In the coordinate ring  $\mathbb{C}[G] = \bigoplus_{\lambda \in \Delta} V(\lambda) \otimes V(\lambda^*)$ , specialization at  $Id$  is contraction of  $V(\lambda)$  against the dual  $V(\lambda^*)$ . The coordinate ring  $\mathbb{C}[U_- \setminus G] \subset \mathbb{C}[G]$  is the subalgebra of spaces  $\mathbb{C}v_{-\eta} \otimes V(\eta)$ , it therefore follows that  $\pi^*(f) \in \mathbb{C}[U_+]$  is the coefficient of the  $v_\eta$  component of  $f$ . Since  $f$  was chosen to have weight  $\mu^*$ ,  $\pi^*(f) \in V_\eta(\lambda)$  must be a  $\mu^* - \eta$  weight vector.

**Lemma 3.7.** *The space  $W(\mu, \lambda, \eta) \cong (V(\lambda) \otimes V(\eta))_{\mu^*}^{U_-}$  is isomorphic to  $V_{\eta, \mu^* - \eta}(\lambda)$ .*

*Proof.* We have already established a  $1-1$  map  $\pi^* : (V(\lambda) \otimes V(\eta))_{\mu^*}^{U_+} \rightarrow V_{\eta, \mu^* - \eta}(\lambda)$ . To show that this map is also onto, we observe that  $(V(\lambda) \otimes V(\eta))^{U_+}$  is a direct sum of its dominant weight spaces, each of which maps to a distinct  $V_{\eta, \mu^* - \eta}(\lambda) \subset V_\eta(\lambda)$ , and that  $(V(\lambda) \otimes V(\eta))^{U_+} \cong V_\eta(\lambda)$ .  $\square$

The torus  $T_1 \times T_2 \times T_3$  acts on the space  $V_{\eta, \mu^* - \eta}(\lambda)$  with character  $((\mu^* - \eta + \eta)^*, \lambda, \eta) = (\mu, \lambda, \eta)$ . We have now established a  $T^3$ -stable map of algebras, identifying  $\mathbb{C}[P_3(G)]$  with the subspace  $\bigoplus_{\mu, \lambda, \eta} V_{\eta, \mu^* - \eta}(\lambda) t^\eta \subset \mathbb{C}[U_- \setminus G \times T]$ .

**Theorem 3.8.** *For each  $\mathbf{i}$  there is a  $T^3$  stable valuation  $v_{\mathbf{i},3}$  on  $\mathbb{C}[P_3(G)]$ , with associated graded ring equal to the affine semigroup algebra  $\mathbb{C}[C_{\mathbf{i}}(3)]$ . The torus  $T_1 \times T_2 \times T_3$  acts on  $\mathbb{C}[C_{\mathbf{i}}(3)]$  with characters  $\pi_1, \pi_2, \pi_3 : C_{\mathbf{i}}(3) \rightarrow \Delta$ . Furthermore,  $v_{\mathbf{i},3}(B_3)$  coincides with the integer points in  $C_{\mathbf{i}}(3)$ .*

*Proof.* The valuation  $v_{\mathbf{i},3}$  is constructed from  $<$  on  $\Delta$  and  $v_{\mathbf{i}}$  on  $\mathbb{C}[U_- \setminus G]$ . It is invariant with respect to  $T_1 \times T_2 \times T_3$ , because  $v_{\mathbf{i}}, < : \mathbb{C}[U_- \setminus G \times T] \rightarrow C_{\mathbf{i}}$  is  $T^4$  invariant. By construction the character with respect to the torus action on  $\mathbb{C}[C_{\mathbf{i}}(3)]$  corresponds to the maps  $\pi_1, \pi_2, \pi_3$ . The image  $v_{\mathbf{i},3}(\mathbb{C}[P_3(G)])$  is then the image of those  $b \otimes t^\eta \in \mathbb{C}[U_- \setminus G \times T]$  which lie in  $\bigoplus_{\mu, \lambda, \eta} V_{\eta, \mu^* - \eta}(\lambda) t^\eta$  under  $v_{\mathbf{i}}, <$ . This coincides with the integer points in  $C_{\mathbf{i}}(3)$  by construction.  $\square$

This exposition has been for the semisimple case, but as in [AB], everything can be generalized readily to the reductive case. The weights that define a non-zero  $W(\mu, \lambda, \eta)$  are of the form  $\mu' + \tau_1$ ,  $\eta' + \tau_2$  and  $\lambda' + \tau_3$  where  $\tau_i$  are characters of the center  $Z(G)$  with  $\tau_1 + \tau_2 + \tau_3 = 0$ , and  $\mu', \eta', \lambda'$  are dominant weights of the semisimple part of  $G$ . The subspace  $V_{\eta, \mu - \eta}(\lambda)$  is the same as the subspace  $V_{\eta', \mu' - \beta' + (\tau_1 - \tau_2)}(\lambda' + \tau_3) = V_{\eta', \mu' - \eta' + \tau_3}(\lambda' + \tau_3) = V_{\eta', \mu' - \eta'}(\lambda') \otimes \mathbb{C}\tau_3$ . So this space inherits the subset of the dual canonical basis of the semi-simple part of  $G$  coming from  $V_{\eta', \mu' - \beta'}(\lambda')$  tensored with the character  $\tau_3$ . Everything else goes through as above after a total order has been chosen on the characters of the center  $\mathfrak{z} \subset \mathfrak{g}$ .

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

We can now construct filtrations on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  and  $\mathbb{C}[P_n(G)]$  with toric associated graded algebras  $\mathbb{C}[C_i(\Gamma)]$ ,  $\mathbb{C}[C_i(\mathcal{T}, \vec{\lambda})]$ . We focus on the filtrations on the algebra  $\mathbb{C}[\mathcal{X}(F_g, G)]$ , we choose  $\Gamma$ , with a total ordering on  $E(\Gamma)$ , a total ordering on  $V(\Gamma)$ , and an assignment  $\mathbf{i} : V(\Gamma) \rightarrow R(w_0)$ .

The efforts of Section 2 give a filtration on  $\mathbb{C}[\mathcal{X}(F_g, G)]$  by  $(\Delta, <)^{E(\Gamma)}$  with associated graded algebra  $\mathbb{C}[P_\Gamma(G)]$ . In Section 3 we construct a  $T^3$  invariant valuation on  $\mathbb{C}[P_3(G)]$  with associated graded algebra  $\mathbb{C}[C_i(3)]$ . We use the ordering on  $V(\Gamma)$  and the assignment  $\mathbf{i} : V(\Gamma) \rightarrow R(w_0)$  to define a full rank,  $T^{E(\Gamma)}$ -stable valuation on  $\bigotimes_{v \in V(\Gamma)} \mathbb{C}[P_3(G)]$  with associated graded algebra  $\bigotimes_{v \in V(\Gamma)} \mathbb{C}[C_{\mathbf{i}(v)}(3)]$ . Passing to  $T^{E(\Gamma)}$  invariants gives a filtration on  $\mathbb{C}[P_\Gamma(G)]$ , and by Theorem 3.8 the associated graded algebra is the semigroup algebra of the following polyhedral cone.

**Definition 4.1.** We let  $\pi_{v,i}$  be the projection map on  $C_{\mathbf{i}(v)}(3)$  defined by the edge  $i$  incident on the vertex  $v \in V(\Gamma)$ .

We define  $C_i(\Gamma)$  to be the toric fiber product cone in  $\prod_{v \in V(\Gamma)} C_{\mathbf{i}(v)}(3)$  defined by the conditions  $\pi_{v,i} = \pi_{u,i}^*$  for all edges  $i$  with endpoints  $u, v$ .

Now Theorem 1.1 follows from Proposition 1.5. The same program can be carried out on the algebra  $\mathbb{C}[P_n(G)]$ , giving a  $T^n$ -stable valuation with associated graded algebra  $\mathbb{C}[C_i(\mathcal{T})]$ . Theorem 1.2 then follows by specializing the weights at the leaves of  $\mathcal{T}$  to  $\vec{\lambda}$ , using Theorem 3.8. The following are also immediate.

**Proposition 4.2.** For every choice of a trivalent tree  $\mathcal{T}$  with  $n$  ordered leaves, and an assignment  $\mathbf{i} : V(\mathcal{T}) \rightarrow R(w_0)$  we have

- (1) A basis  $B(\mathcal{T}, m\vec{\lambda}) \subset H^0(P_{\vec{\lambda}^*}(G), \mathcal{L}(m\vec{\lambda}^*)) = (V(\lambda_1) \otimes \dots \otimes V(\lambda_n))^G$ ,
- (2) A labelling  $v_{\mathcal{T}, \mathbf{i}} : B(\mathcal{T}, m\vec{\lambda}) \rightarrow C_i(\mathcal{T}, m\vec{\lambda}) \subset (\Delta \times \mathbb{Z}_{\geq 0}^N \times \Delta)^{V(\mathcal{T})}$ .

We conclude by remarking that the image of the valuations we've constructed coincide with all of the lattice points in their corresponding convex bodies.

**Proposition 4.3.** The integer points in  $C_i(\Gamma)$  (resp.  $C_i(\mathcal{T})$ ) are in bijection with the images of the induced valuations  $v_{\mathbf{i}, \Gamma}$  and  $v_{\mathbf{i}, \mathcal{T}}$ , respectively.

*Proof.* Each basis member of  $\mathbb{C}[\mathcal{X}(F_g, G)]$  gives an element of  $C_i(\Gamma)$  by the constructions in Sections 2, 3. If  $\vec{t} \in C_i(\Gamma)$  is an integer point, it is likewise an integer point in  $(\Delta \times \mathbb{Z}_{\geq 0}^N \times \Delta)^{V(\Gamma)}$ , and therefore corresponds to a product  $\bigotimes_{v \in V(\Gamma)} b_v$  of dual canonical basis elements with compatible dominant weight data, is in  $B(\Gamma)$  by construction.  $\square$

## 5. EXAMPLES

We describe the cones  $C_{\mathbf{i}}(3)$  for  $G = SL_m(\mathbb{C})$  and all rank 2 simple groups. For  $G = SL_m(\mathbb{C})$  we describe particular instances of  $B_{\mathbf{i}}(\Gamma), B_{\mathbf{i}}(\mathcal{T})$ . The inequalities we present are culled from both [BZ2] and the treatment by Littelman [Li].

**5.1. Type A.** For  $G = SL_m(\mathbb{C})$ , we take  $\mathbf{i}$  to be the "nice" decomposition (see [Li]).

$$(18) \quad w_0 = s_1(s_2s_1)(s_3s_2s_1)\dots(s_{m-1}\dots s_1)$$

The polyhedron  $C_{\mathbf{i}}(3)$  is then the cone  $BZ_3(SL_m(\mathbb{C}))$  of Berenstein-Zelevinsky triangles [BZ3], for more on these objects see [MZ].

**Definition 5.1.** *For this definition we refer to Figure 5. A BZ triangle  $T \in BZ_3(SL_m(\mathbb{C}))$  is an assignment of non-negative integers to vertices of the version of the diagram in Figure 5 with  $2(m-1)$  vertices on a side. If  $v, w$  are a pair of vertices which are across a hexagon from a pair  $u, y$ , then  $T(v) + T(w) = T(u) + T(y)$ .*

*We let  $a_1, \dots, a_{2m-2} = b_1, \dots, b_{2m-2} = c_1, \dots, c_{2m-2} = a_1$  label the vertices clockwise around the boundary of the diagram. This lets us define the following three projection maps  $\pi_1, \pi_2, \pi_3 : BZ_3(SL_m(\mathbb{C})) \rightarrow \Delta_{SL_m(\mathbb{C})}$ .*

$$(19) \quad \pi_1(T) = (a_1 + a_2, \dots, a_{2m-3} + a_{2m-2})$$

$$\pi_2(T) = (b_1 + b_2, \dots, b_{2m-3} + b_{2m-2})$$

$$\pi_3(T) = (c_1 + c_2, \dots, c_{2m-3} + c_{2m-2})$$

The maps  $\pi_i$  are constructed to coincide with their counterparts in Section 3.

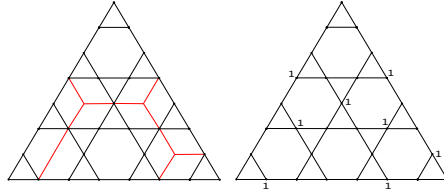


FIGURE 5. Left: A honeycomb weighting. Right: Dual weighting by non-negative integers.

We can associate a dual graph to each  $T \in BZ_3(SL_m(\mathbb{C}))$  by replacing each entry of weight  $a$  with an edge to the center of its adjacent hexagon weighted  $a$ . The resulting graphs are also called honeycombs, and have been studied by a number of authors, see e.g. [KTW], [GP].



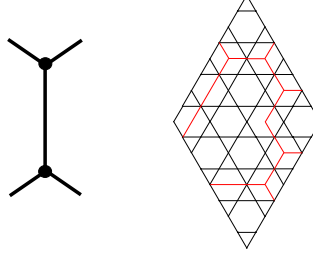


FIGURE 6. A BZ quilt with dual 4-tree.

The cone of  $\Gamma$ -BZ quilts  $BZ_\Gamma(SL_m(\mathbb{C})) \subset \prod_{v \in V(\Gamma)} BZ_3(SL_m(\mathbb{C}))$  is then defined to be those tuples  $(T_v)$  with  $\pi_{i,v}(T_v) = \pi_{i,u}(T_u)^*$  when an edge  $i$  joins  $u$  and  $v$ , see Figures 6 and 7.

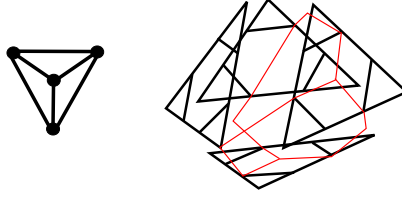


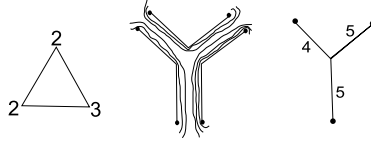
FIGURE 7. A BZ quilt with dual genus 3 graph.

We represent gluing triangles  $T_1, T_2$  with matching boundary components as weighted graphs on composite diagrams, as in Figure 6. Paths at the meeting boundaries of  $T_1, T_2$  are joined by an arrangement of weighted paths in an  $X$  configuration. When the path has weight 1, this is either a line (see the left corner of the quilt in Figure 6) or by a crooked line (see the right corner of the quilt in Figure 6). The cones  $BZ_\Gamma(SL_3(\mathbb{C}))$  are studied by the author and Zhou in [MZ].

We finish this subsection with an analysis of the generators of the semigroup algebra  $\mathbb{C}[BZ_\Gamma(SL_2(\mathbb{C}))]$ . The semigroup  $BZ_3(SL_2(\mathbb{C}))$  is the free semigroup on three generators, we depict an element of this semigroup as an arrangement of three types of paths in a dual trinode  $\tau$ , see Figure 8.

For an element  $T \in BZ_3(SL_2(\mathbb{C}))$ , counting the number of endpoints in each edge  $e, f, g$  of  $\tau$  produces an integer weighting  $w_T : \{e, f, g\} \rightarrow \mathbb{Z}_{\geq 0}$ . These three numbers must satisfy the triangle inequalities,  $|w_T(e) - w_T(f)| \leq w_T(g) \leq w_T(e) + w_T(f)$ , and  $w_T(e) + w_T(f) + w_T(g) \in 2\mathbb{Z}$ .

The semigroup  $BZ_\Gamma(SL_2(\mathbb{C}))$  can then be described as the set of weightings  $w : E(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$  which satisfy these properties at each  $v \in V(\Gamma)$ . We associate a

FIGURE 8. Three different interpretations of an element of  $BZ_3(SL_2(\mathbb{C}))$ .

planar arrangement of paths  $P(w)$  in  $\Gamma$ , by replacing the weights at each trinode with an arrangement of paths as above. For an edge  $e \in E(\Gamma)$ , the endpoints of these paths in  $e$  are then connected in the unique planar way. Symmetrically, a path  $\gamma$  in the graph  $\Gamma$  has an associated weighting  $w_\gamma : E(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$  obtained by setting  $w_\gamma(e)$  equal to the number of times  $\gamma$  passes through  $e$ . For a more in depth account of this construction, see [M2].

**Theorem 5.2.** *The semigroup  $BZ_\Gamma(SL_2(\mathbb{C}))$  is generated by the  $w : E(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$  with  $w(e) \leq 2$ .*

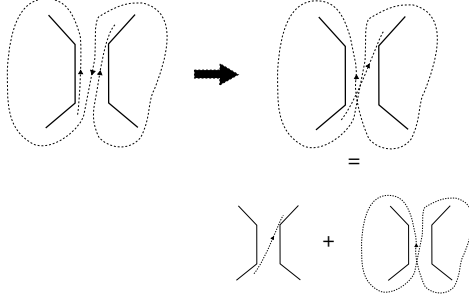


FIGURE 9. Using an orientation to factor off a loop.

*Proof.* Fix a  $w \in BZ_\Gamma(SL_2(\mathbb{C}))$ , and consider the induced planar arrangement of paths  $P(w)$  in  $\Gamma$  with multiweight  $w$ . Suppose  $w(e) > 2$  for some edge  $e \in E(\Gamma)$ . We pick a path  $\gamma \in P(w)$  which passes through  $e$ . If  $\gamma$  passes through  $e$  with weight 1, then we may remove  $w_\gamma$  to obtain a weighting  $w'$  with strictly smaller total weight  $\sum_{f \in E(\Gamma)} w'(f)$ .

If  $\gamma$  weights  $e$  greater than 1, we assign an orientation to  $\gamma$ . If two components at  $e$  have the same direction, we may alter the weighting as in Figure 9, yielding two closed paths  $\gamma' \cup \gamma''$ . Without loss of generality we assume that  $P(w) = \{\gamma\}$ , so that the weightings  $w_{\gamma'}$  and  $w_{\gamma''}$  satisfy  $w_{\gamma'} + w_{\gamma''} = w$ . In this case we pull off the new closed path  $\gamma'$ , which has strictly smaller total weight. If  $w(e) > 2$  at least two components through  $e$  must have the same direction.  $\square$

As a corollary, a set of functions in  $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$  which represent the set of  $w \in BZ_\Gamma(SL_2(\mathbb{C}))$  with  $w(e) \leq 2$  form a finite subduction basis for the filtration defined by  $\Gamma$ .

5.2.  $G_2$ . We give inequalities for the tensor product cones for  $G_2$ , for  $R(w_0) = \{\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\}$ . These cones have six string parameters  $t_1, t_2, t_3, t_4, t_5, t_6$ , and six weight parameters  $\lambda = (\lambda_1, \lambda_2), \eta = (\eta_1, \eta_2), \mu = (\mu_1, \mu_2)$ . The cone  $C_{\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2}(3)$  is defined by the following inequalities.

$$(20) \quad 6t_2 \geq 2t_3 \geq 3t_4 \geq 2t_5 \geq 6t_6 \geq 0; \quad \lambda_2 \geq 2t_6$$

$$(21) \quad \eta_1 \geq t_1 - 3t_2 - t_3 - 3t_4 - t_5 - 3t_6, \quad t_3 - 3t_4 - t_5 - 3t_6, \quad t_5 - 3t_6$$

$$(22) \quad \eta_2 \geq t_6, \quad t_4 - t_5 - 3t_6, \quad t_2 - t_3 - 3t_4 - t_5 - 3t_6$$

$$(23) \quad 2t_1 - 3t_2 + 2t_3 - 3t_4 + 2t_5 - 3t_6 = \lambda_1 + \eta_1 - \mu_1$$

$$-t_1 + 2t_2 - t_3 + 2t_4 - t_5 + 2t_6 = \lambda_2 + \eta_2 - \mu_2$$

The alternative cone  $C_{\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1}(3)$  is defined by the following inequalities.

$$(24) \quad 2t_2 \geq 2t_3 \geq t_4 \geq 2t_5 \geq 2t_6 \geq 0$$

$$(25) \quad \lambda_1 \geq t_6; \quad \lambda_2 \geq t_2 + t_4 - t_5, \quad t_2 + t_5 - t_6, \quad t_3 - t_4 - t_6, \quad t_5 - 3t_6,$$

$$t_2 + t_3 - 2t_4, \quad 2t_2 - t_4, \quad 3t_2 - t_3, \quad t_3 - t_5, \quad t_4 - 2t_6, \quad 2t_4 - t_5 - t_6, \quad 3t_4 - 2t_5$$

$$(26) \quad \eta_1 \geq t_6, \quad t_4 - 3t_5 - t_6, \quad t_2 - 3t_3 - t_4 - 3t_5 - t_6$$

$$(27) \quad \eta_2 \geq t_5 - t_6, \quad t_3 - t_4 - 3t_5 - t_6, \quad t_1 - t_2 - 3t_3 - t_4 - 3t_5 - t_6$$

$$(28) \quad 2(t_2 + t_4 + t_6) - 3(t_1 + t_3 + t_5) = \lambda_1 + \eta_1 - \mu_1$$

$$2(t_1 + t_3 + t_5) - (t_2 + t_4 + t_6) = \lambda_2 + \eta_2 - \mu_2$$

5.3.  $SP_4$ . We give the inequalities for the  $SP_4$  tensor product cones corresponding to the decompositions  $R(w_0) = \{\alpha_1\alpha_2\alpha_1\alpha_2, \alpha_2\alpha_1\alpha_2\alpha_1\}$ . There are four string parameters  $t_1, t_2, t_3, t_4$ , and dominant weight parameters  $\lambda = (\lambda_1, \lambda_2), \eta = (\eta_1, \eta_2), \mu = (\mu_1, \mu_2)$ . The cone  $C_{\alpha_1\alpha_2\alpha_1\alpha_2}(3)$  is defined by the following inequalities.

$$(29) \quad 2t_2 \geq t_3 \geq 2t_4 \geq 0$$

$$(30) \quad \lambda_2 \geq t_4; \quad \lambda_1 \geq 2t_3 - 2t_4, \quad 2t_2 - 2t_3 - 2t_4, \quad 2t_1 + 2t_2$$

$$(31) \quad \eta_1 \geq t_1 - t_2 + 2t_3 - t_4, \quad t_3 - t_4; \quad \eta_2 \geq t_2 - 2t_3 + 2t_4, \quad t_4$$

$$(32) \quad 2t_1 - 2t_2 + 2t_3 - 2t_4 = \lambda_1 + \eta_1 - \mu_1$$

$$-t_1 + 2t_2 - t_3 + 2t_4 = \lambda_2 + \eta_2 - \mu_2$$

The cone  $C_{\alpha_2\alpha_1\alpha_2\alpha_1}(3)$  is defined by the following inequalities.

$$(33) \quad t_2 \geq t_3 \geq t_4 \geq 0$$

$$(34) \quad \lambda_2 \geq 2t_1, t_2; \quad \lambda_1 \geq 2t_1$$

$$(35) \quad \eta_1 \geq t_3 - t_4, \quad t_1 - t_2 + 2t_3; \quad \eta_2 \geq t_2 - 2t_3 + 2t_4, \quad t_4$$

$$(36) \quad t_1 + t_2 + t_3 + t_4 = \lambda_1 + \eta_1 - \mu_1$$

$$t_2 + t_4 - t_1 - t_3 = \lambda_2 + \eta_2 - \mu_2$$

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